2022

(November)

MATHEMATICS

(Elective/Honours)

(GHS–11)

(Algebra—I and Calculus—I)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** question from each Unit

UNIT—I

1. *(a)* Find the domain and range of the function

$$f(x)=\frac{|x|}{x}$$ 1+1=2

(b) If

$$f(x)=b\cdot\frac{x-a}{b-a}+a\cdot\frac{x-b}{a-b}$$

then show that $f(a)+f(b)=f(a+b)$. 2

(c) Examine the continuity of the function

$$f(x) = \frac{x^4 + 2x^3 + 2x}{\sin x}, \quad x \neq 0$$
$$= 0 \quad , \quad x = 0$$

at the point $x = 0$. 3

(d) Let $\mathbb{Z}$ be the set of all integers and a relation R on $\mathbb{Z}$ is defined as $R = \{(a, b) : a - b \text{ is divisible by } 2\}$. Show that R is an equivalance relation. 3

(e) Show that

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = 1$$ 3

(f) Draw the graph of the function $f(x) = x - [x]$, where $[x]$ denotes the greatest integer not greater than x. 2

2. (a) If a set A has n elements, what is the number of elements of $P(A)$, where $P(A)$ stands for the power set of A? Justify your answer. 2

(b) Prove that for any two sets A and B, $(A \cap B)^c = A^c \cup B^c$; A^c denotes the complement of A. 3

(c) A function $f : \mathbb{R} - \{1\} \to \mathbb{R}$ is defined by

$$f(x) = \frac{x+1}{x-1}$$

Show that f is one-one but not onto. 3

(d) If $f(x) = x^2 - 5x + 6$, then find $f(x+1)$. 2

(e) For what value of k is the function defined by

$$f(x) = \frac{\sin kx}{x}, \quad x \neq 0$$
$$= 2k - 3, \quad x = 0$$

continuous at $x = 0$? 3

(f) Give an example of a relation which is—

(i) symmetric but not transitive;

(ii) reflexive and antisymmetric. 1+1=2

UNIT—II

3. (a) Show that the matrix

$$\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

is skew-symmetric. 2

(b) Prove that every square matrix can be uniquely expressed as $P + iQ$, where P and Q are Hermitian matrices. 5

(c) Reduce the matrix

$$M = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

to normal form and determine its rank. 6

(d) If A be an $n \times n$ matrix, then prove that $|\text{adj } A| = |A|^{n-1}$. 2

4. (a) Show that the matrix

$$P = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

is nilpotent and state its index. 2+1=3

(b) Examine the consistency of the following system of equations :

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

If consistent, then solve the system. 4+2=6

(c) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

by using elementary operations. 6

UNIT—III

5. (a) State two properties of continuous function defined on a closed and bounded interval. 4

(b) Evaluate $\frac{dy}{dx}$ of the following (any *two*) : 3×2=6

(i) $x^y = y^x$

(ii) $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

(iii) $\tan x = \frac{2t}{1-t^2}$, $\sin y = \frac{2t}{1+t^2}$

(c) Find the equation of the tangent to the curve $y = x^2 + 3x + 5$ at the point $(1, -1)$. 3

(d) Differentiate $\tan^{-1} x$ with respect to x^2. 2

6. (a) If the side of an equilateral triangle increases at the rate of $\sqrt{3}$ cm per second and its area at the rate of 12 cm^2 per second, then find the length of the side of the triangle. 3

(b) If $\log y = \tan^{-1} x$, then prove that

(i) $(1 + x^2)y_2 + (2x - 1)y_1 = 0$

(ii) $(1 + x^2)y_{n+2} + (2nx - 2x - 1)y_{n+1} + n(n+1)y_n = 0$ 2+4=6

(c) Evaluate the following using L'Hospital's rule (any *two*) : 3×2=6

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{\log x^2}{\log(\cot^2 x)}$

(ii) $\underset{x\to 0}{\text{Lt}} \dfrac{x^3}{e^x}$

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^{2x}-1}{x}$

UNIT—IV

7. (a) Evaluate (any *two*) : 3×2=6

(i) $\int \dfrac{1}{1+\cot x}dx$

(ii) $\int \dfrac{x^2}{x^2+9}dx$

(iii) $\int \dfrac{x}{\sqrt{x}+1}dx$

(b) Show that

$$\int_0^{\pi/4} \log(1+\tan\theta)d\theta = \frac{\pi}{8}\log 2$$ 6

(c) Evaluate by the method of summation 3

$$\int_0^1 (x^2+1)dx$$

8. (a) Evaluate : 4

$$\underset{n\to\infty}{\text{Lt}}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{1}{2n}\right]$$

(b) Show that

$$I_n = \int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

where n is a positive integer. Hence, evaluate

$$\int_0^{\pi/4} \tan^4 x\,dx$$ 4+2=6

(c) Evaluate

$$\int_3^{\infty} \frac{1}{(x-2)^2}dx$$

and discuss its convergence. 3

(d) Show that

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx = \frac{\pi}{4}$$ 2

UNIT—V

9. (a) Find the differential equation of the family of curves $y = e^x(A\cos x + B\sin x)$ where A and B are constants. 3

(b) Solve any *three* of the following : 3×3=9

(i) $(xy^2 + x)dx + (yx^2 + y)dy = 0$

(ii) $\cos x \dfrac{dy}{dx} + y \sin x = \sec^2 x$

(iii) $\dfrac{dy}{dx} = (x + y)^2$

(iv) $x^2 y dx - (x^3 + y^3) dy = 0$

(c) Show that the equation

$$(ax + hy + g)dx + (hx + by + f)dy = 0$$

is exact and solve it. 3

10. *(a)* Solve any *two* of the following : 4×2=8

(i) $p^2 + p - 6 = 0$

(ii) $p^2 - 2xp + 1 = 0$

(iii) $p^2 - p(x + y) + xy = 0$

Here p stands for $\dfrac{dy}{dx}$.

(b) Find the orthogonal trajectories of the curve $x^2 + y^2 + 2gx + c = 0$, where g is a parameter. 4

(c) Reduce the equation $x^2(y - px) = p^2 y$ to Clairaut's form and solve it. 3

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