3/EH–29 (iii) (Syllabus–2015)

2022

(November)

MATHEMATICS

(Elective/Honours)

(GHS–31)

(Algebra—II and Calculus—II)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

Unit—I

1. (a) Prove that the set $\mathbf{C}$ of all complex numbers $z = a + ib$; $a, b \in \mathbb{R}$ forms an infinite Abelian group with respect to addition of complex numbers. 5

(b) Prove that the additive group $(\{0, 1, 2, 3, 4\}, +_5)$ is cyclic. Also, find its generators. 4+1=5

(c) Let a be an element of a group G. Show that the set $H = \{a^n : n \in I\}$ of all integral powers of a is a subgroup of G. 5

2. (a) Show that every group of prime order p is cyclic. Is it Abelian? 4+1=5

(b) State and prove Lagrange's theorem on the order of a finite group. 1+4=5

(c) Let H be a subgroup of G and $T = \{x : x \in G \text{ and } xH = Hx\}$. Prove that T is a subgroup of G. 5

UNIT—II

3. (a) Solve the equation

$$x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$$

given that one of its roots is $2 + \sqrt{3}$. 5

(b) Solve the equation $x^3 + 63x - 316 = 0$ by Cardan's method. 6

(c) Solve the equation $x^3 - 7x^2 + 36 = 0$ given that one root is double of another. 4

4. (a) Apply Descarte's rule of signs to discuss the nature of the roots of the equation $x^4 + 15x^2 + 7x - 11 = 0$. 4

(b) Find the equation whose roots are the roots of $3x^3 - 2x^2 + x - 9 = 0$ each diminished by 5. 4

(c) Let α, β, γ and δ be the roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

Find the values of the following symmetric functions : 1+3+3=7

(i) $\Sigma\alpha$

(ii) $\Sigma\alpha^2\beta$

(iii) $\Sigma\alpha^2\beta\gamma$

UNIT—III

5. (a) Prove that if a sequence converges, then its limit is unique. 4

(b) Prove that the sequence $\{(-1)^n\}$ is not a Cauchy sequence. 5

(c) Prove that the sequence $\left\{\frac{4n+3}{n+2}\right\}$ is bounded and monotonically increasing. 2+2=4

(d) Show that the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ is convergent. 2

6. (a) Test the convergence of any *two* of the following series : 3×2=6

(i) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$

(ii) $\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}}\right)$

(iii) $\sum_{n=2}^{\infty} \frac{1}{\log n}$

(b) What is an alternating series? State Leibnitz's test for the convergence of an alternating series and hence show that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

converge. 1+2+3=6

(c) Define radius of convergence of a series. Find the interval of convergence of the series $1+x+2!\,x^2+3!\,x^3+\cdots$. 1+2=3

UNIT—IV

7. (a) State and prove Rolle's theorem. Also, give its geometrical interpretation. 1+3+2=6

(b) Find the maximum value of $\left(\frac{1}{x}\right)^x$. 4

(c) Show that the radius of curvature at $\theta = \frac{\pi}{4}$ on the curve $x = a\cos^3\theta$; $y = a\sin^3\theta$ is $\frac{3a}{2}$. 5

8. (a) Find the horizontal and vertical asymptotes, (if any) of the curve $y^2(x^2-a^2)=x$. 5

(b) For the function

$$f(x, y)=\frac{x^2y^2}{x^2y^2+(x-y)^2},$$

show that

$$\underset{x\to 0}{\text{Lt}}\ \underset{y\to 0}{\text{Lt}}\ f(x, y)=\underset{y\to 0}{\text{Lt}}\ \underset{x\to 0}{\text{Lt}}\ f(x, y)$$

but $\underset{\substack{x\to 0\\ y\to 0}}{\text{Lt}}\ f(x, y)$ does not exist. 5

(c) If $u=\sin^{-1}\frac{x^2+y^2}{x+y}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$. 5

UNIT—V

9. (a) State and prove the fundamental theorem of integral calculus. 1+5=6

(b) Expand $f(x)=\log(1+x)$ in a finite series in powers of x with remainder in Lagrange's form. 5

(c) Apply the method of double integration to find the area of a quadrant of the ellipse $9x^2+16y^2=144$. 4

10. (a) Find the length of the arc of the parabola $y^2=4ax$ intercepted between the vertex and an extremity of the latus rectum. 5

(b) Find the volume of the solid generated by revolution of the circle $x^2+y^2=a^2$ about x-axis. 5

(c) Evaluate

$$\int_0^3\int_1^{\sqrt{4-y}}(x+y)\,dx\,dy$$

by changing the order of integration. 5

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