2022

(November)

MATHEMATICS

(Honours)

(H–52)

(Advanced Calculus—I)

Marks : 45

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **three** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* Show that a bounded function f is integrable if and only if to every $\varepsilon > 0$, there corresponds a $\delta > 0$, such that for every division *D*, whose norm is $\leq \delta$, the oscillatory sum $w(D)$ is $< \varepsilon$. 7

(b) Show that the integrability of $|f|$ may not imply the integrability of function f. 3

(c) By applying the first mean value theorem of integral calculus, show that

$$\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \frac{1}{\sqrt{1-\frac{k^2}{4}}}$$ 5

2. (a) Show that the Gamma function

$$\int_0^\infty x^{n-1}e^{-x}dx$$

is convergent if and only if $n > 0$. 5

(b) If ϕ is continuous in $[0, \infty[$ and $\lim_{x\to 0} \phi(x) = \phi_0$ and $\lim_{x\to\infty} \phi(x) = \phi_1$, then show that

$$\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x} dx = (\phi_0 - \phi_1) \times \log\left(\frac{b}{a}\right)$$ 6

(c) Prove that

$$\int_1^\infty \frac{\sin x}{x^p} dx$$

is convergent for $p > 0$. 4

UNIT—II

3. (a) Show that

$$\int_0^{\pi/2} \log(1 - x^2 \sin^2\theta)\, d\theta = \pi \log(1 + \sqrt{1-x^2}) - \pi \log 2$$

if $|x| < 1$. 7

(b) When do you say that an improper integral

$$\int_a^\infty f(x, y) dx$$

converges uniformly on an interval? 3

(c) Evaluate : 5

$$f(y) = \int_0^\infty \frac{\cos xy}{1+x^2} dx$$

4. (a) Show that

$$\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a)$$

if $a \ge 0$. 5

(b) If f is a continuous function of two variables with rectangle $[a, b; c, d] \subset \mathbb{R}^2$ as its domain, then show that the function

$$\phi(y) = \int_a^b f(x, y) dx$$

is continuous in $[c, d]$. 4

(c) Starting from a suitable integral, show that

$$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3} \tan^{-1}\frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}$$ 6

UNIT—III

5. *(a)* Use Green's theorem to compute the difference between the line integrals

$$I_1 = \int_{ACB} \{(x+y)^2 dx - (x-y)^2 dy\}$$

and $I_2 = \int_{ADB} \{(x+y)^2 dx - (x-y)^2 dy\}$

where ACB and ADB are respectively the straight line $y = x$ and the parabolic arc $y = x^2$, joining the points $A(0, 0)$ and $B(1, 1)$. 7

(b) Change the order of integration in

$$\int_0^{2a} \int_{x^2/4a}^{3a-x} \phi(x, y) dx\, dy$$ 5

(c) Evaluate

$$\iint (x^2 + y^2) dx\, dy$$

over the circle $x^2 + y^2 = a^2$. 3

6. *(a)* State the Stokes' theorem and use it to find the line integral

$$\int_C (x^2 y^3 dx + dy + z dz)$$

where C is the circle $x^2 + y^2 = a^2$, $z = 0$. 2+6=8

(b) Show that

$$\int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{x^2 + y^2} dy \right\} dx = \int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{x^2 + y^2} dx \right\} dy$$ 7

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