5/H–29 (v) (Syllabus–2019)

2022

(November)

MATHEMATICS

(Honours)

(H–51)

(Elementary Number Theory)

Marks : 30

Time : 2 hours

The figures in the margin indicate full marks for the questions

Answer **two** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* State whether the following statements are true or false giving brief justifications (any *five*) : 2×5=10

(*a*, *b*, *c* and *n* denote integers)

(i) If $(a, b) = (a, c)$, then $[a, b] = [a, c]$.

(ii) If $a^3 \mid c^3$, then $a \mid c$.

(iii) If a is odd, then $(a, a+2) = 1$.

(iv) $n^4 + 4$ is composite $\forall\, n > 1$.

(v) If $(a, 6) = 2$ and $(b, 6) = 3$, then $(a + b, 6) = 2$.

(vi) There exists an integer n such that $31 \mid n^2 - 30$.

(b) Show that $\underline{|61} + 1$ is divisible by 71. 3

(c) Show that the set $1^2, 2^2, \cdots, m^2$ is not a complete residue system modulo m if $m > 2$. 2

2. *(a)* State and prove Wilson's theorem. 1+4=5

(b) If p is an odd prime, then show that

$$1^2 \cdot 3^2 \cdot 5^2 \ldots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod p$$ 3

(c) What is the last digit in the ordinary decimal representation of 6^{200}? 3

(d) If n is odd, then show that

$$8 \mid 5n^9 + 2n^5 + n^3$$ 2

(e) Express the g.c.d. of 72 and 45 in the form $72x + 45y$, where x and y are integers. 2

UNIT—II

3. *(a)* Find the smallest positive integer x that satisfies

$$x \equiv 3 \pmod 5$$
$$x \equiv 1 \pmod 7$$
$$x \equiv 5 \pmod 8$$ 4

(b) Solve the congruence $24x \equiv 40 \pmod{56}$. 3

(c) If m, n are positive integers with $(m, n) = 1$, then prove that

$$\phi(mn) = \phi(m)\,\phi(n)$$ 5

(d) Find the number of multiples of 14 between 120 and 1200. 3

4. *(a)* Find the highest power of 12 dividing $\underline{|620}$. 3

(b) Prove that $\mu(n)$ is multiplicative and

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$ 3

(c) Show that $\sum_{d|n} \mu(d)\tau(d) = (-1)^{\omega(n)}$. 3

(d) Evaluate $\sigma(450)$ and $\tau(1815)$. 2

(e) For what real number x is it true that $[9x] = 9$? Justify your answer. 2

(f) Show that $\phi(2^k) = 2^{k-1}$. 2

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