2023

(May/June)

MATHEMATICS

(Elective/Honours)

(Algebra—II and Dynamics)

(GHS–41)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

Use two separate answer books for Algebra—II and Dynamics

(Algebra—II)

UNIT—I

1. *(a)* If W is a group, $p, q, r \in W$ and e is the identity element of W, prove the following results : 2×2=4

   (i) $qp = qr \Rightarrow p = r$

   (ii) $pq = e \Rightarrow qp = e$

(b) State the definition of subgroup of a group. If G is a group, $a \in G$ such that $a^n = e$, for some $n \in \mathbb{N}$, then prove the following : 1+3+3=7

(i) $a, a^2, a^3, \ldots, a^{n-1}$ are distinct elements of G.

(ii) $\langle a \rangle = \{a^i : i = 0, 1, 2, \ldots, n-1\}$, where $a^n = e$, for some $n \in \mathbb{N}$, is a subgroup of G.

(c) If A is an Abelian group, prove that $(xy)^m = x^m y^m$, $\forall\, x, y \in A$ and $m \in \mathbb{N}$. 4

2. *(a)* Give a standard definition of a binary operation. If $A = \{a, b\}$ and $f = \{((a, a), a), ((a, b), b)\}$, is f a binary operation on A? Justify your answer. 2+2=4

(b) Prove that any subgroup of a cyclic group is cyclic. Hence deduce that any subgroup of $\mathbb{Z}$ w.r.t. usual addition of integers, is of the form $n\mathbb{Z}$, where $n \in \mathbb{Z}$. 4+2=6

(c) Give a definition of the Euler's phi function, $\phi(n)$. Find the value of $\phi(24)$. 1+1=2

(d) Use Euler's theorem to find the last digit of 7^{20}. 3

UNIT—II

3. *(a)* If $f(x+2) = x^4 - 3x^3 + 4x^2 - 5x - 9$, express $f(x)$ in powers of x. 4

(b) If the roots of the equation $x^n = 1$ are $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, prove that

$$(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1}) = n$$ 3

(c) Solve the equation $x^3 - 7x^2 + 11x + 3$, if one root is $\sqrt{5} - 2$. 4

(d) State Descartes rule of sign. Use it to establish that the equation $x^6 - 3x^2 - x + 1 = 0$ has at least a pair of imaginary roots. 1+3=4

4. *(a)* If α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\Sigma\alpha^2\beta$ in terms of the coefficients. 4

(b) Solve the equation

$$x^4 + x^3 - 16x^2 - 4x + 48$$

given that the product of two of its roots is 6. 5

(c) Solve the equation $x^3 - 6x - 4 = 0$ by Cardan's method. 6

(Dynamics)

UNIT—III

5. (a) A particle of mass m is acted upon by a force $m\mu\left[x+\frac{a^4}{x^2}\right]$ towards the origin. If it starts from rest at a distance a, show that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$. 7

(b) Two smooth spheres of masses m_1, m_2 collide against each other and in the process their velocities change from u_1, u_2 to v_1, v_2, respectively. Show that the loss in kinetic energy is equal to

$$\frac{1}{2}(1-e^2)\frac{m_1 m_2}{m_1+m_2}(u_1-u_2)^2$$

Hence deduce that kinetic energy is conserved for a perfectly elastic impact. 7+1=8

6. (a) A particle of mass m moves in a straight line under the action of a force which is always directed towards a fixed point and which varies inversely as the square of the distance of the particle from the fixed point. If the particle

starts from rest at a distance a from the fixed point, show that the square of the time taken by the particle to reach the fixed point is directly proportional to a^3. 7

(b) A particle moves with SHM in a straight line. In the first second, after starting from rest it travels a distance a and in the next second, it travels a distance b in the same direction. Prove that the amplitude of motion is $\frac{2a^2}{3a-b}$ and its period is $\frac{2\pi}{\cos^{-1}\left(\frac{b-a}{2a}\right)}$. 8

UNIT—IV

7. (a) A body is projected at an angle α to the horizon, so as just to clear two walls of equal height a at a distance $2a$ from each other. Show that the range is equal to $2a\cot\left(\frac{\alpha}{2}\right)$. 7

(b) A particle of mass m moves from rest in a straight line under the action of a constant force in a medium whose resistance to the motion is $m(a+bv)$, where a and b are constants and v is the velocity at time t. If V is the terminal velocity, prove that the particle in time t has moved a distance x, where $bx = V(bt - 1 + e^{-bt})$. 8

8. *(a)* Two bodies are projected from the same point in directions making angles θ_1 and θ_2 with the horizon and strike at the same point in the horizontal plane through the point of projection. If t_1 and t_2 are the corresponding times of flight, show that

$$(t_1^2 - t_2^2)\sin(\theta_1 + \theta_2) = (t_1^2 + t_2^2)\sin(\theta_1 - \theta_2)$$ 7

(b) A ball is thrown vertically upwards in a medium which offers a resistance kv per unit mass when the speed is v. If v_o is the velocity of projection and t_1 is the time that the ball returns to the starting point, prove that $(g + kv_0)(1 - e^{-kt_1}) = gkt_1$. 8

UNIT—V

9. *(a)* A particle describes a curve for which s and ψ vanish simultaneously. If the particle moves with uniform speed u and the acceleration at any point s is $\frac{u^2c}{s^2 + c^2}$, find the intrinsic equation of the curve. 4

(b) A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension in the string has values mW and nW, respectively, when the particle is at the highest and the lowest points of its path. Show that $n = m + 6$. 5

(c) A body describes a circle of radius a with a uniform speed v. Show that the radial and transverse accelerations are $-\frac{v^2}{a}\cos\theta$ and $-\frac{v^2}{a}\sin\theta$, if a diameter is taken as the initial line and one end of the diameter as pole. 6

10. *(a)* A shell of mass m is ejected from a gun of mass M by an explosion which generates kinetic energy E. Prove that the initial velocity of the shell is $\sqrt{\frac{2ME}{(M+m)m}}$ and the recoil velocity of the gun is $\sqrt{\frac{2mE}{M(M+m)}}$. 4

(b) A shell of mass $m_1 + m_2$ moving with velocity u, breaks up into two masses m_1 and m_2, which moves in the same

direction with relative velocity V. Show that the energy of explosion is given by

$$E = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}V^2$$ 5

(c) Show that for a particle, sliding down the arc of a smooth cycloid whose axis is vertical and vertex lowest and starting from a cusp, the vertical velocity is maximum when it has described half the vertical height. 6

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