2 0 2 3

(May/June)

MATHEMATICS

(Honours)

(Advanced Algebra)

(H–62)

Marks : 45

Time : 2 hours

The figures in the margin indicate full marks for the questions

Answer **three** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Prove that every subgroup of an Abelian group is normal. 3

   (b) Give example of a group G and a subgroup H of G such that the index of H in G is equal to 2. Is it true that if N is a normal subgroup of G, then the index of N in G must be equal to 2? Justify your answer. 2+2=4

(Turn Over)

(c) Let G be a group and $g \in G$. Prove that $T_g : G \to G$ defined as

$$T_g(x) = gxg^{-1}, \ \forall x \in G$$

is an automorphism of G. 8

2. *(a)* Prove that a field has no non-trivial ideal. Give an example of a commutative ring which has at least one non-trivial ideal. 4+1=5

(b) Prove that every finite integral domain is a field. Is the converse true? 6+1=7

(c) Prove that $\mathbb{Z} = 36\mathbb{Z} + 55\mathbb{Z}$. 3

UNIT—II

3. *(a)* State and prove the fundamental theorem of ring homomorphism. 2+8=10

(b) Prove that any ideal of a Euclidean ring is a principal ideal. 5

4. *(a)* Prove that if F is a field, then $F[x]$ is an integral domain. 4

(b) Prove that if R is a commutative ring with identity and M is a maximal ideal of R, then R/M is a field. 6

(c) Give the definition of a principal ideal domain (PID). Prove that $\mathbb{Z}_p$ is a PID, where p is a prime. 1+4=5

UNIT—III

5. *(a)* Show that the set of all 2×2 matrices is a vector space over $\mathbb{R}$ with respect to usual addition and scalar multiplication of matrices. 5

(b) Show that the set

$$B = \{1, (x-1), (x-1)^2, (x-1)^3\}$$

forms a basis of the vector space $F_3[x]$ over the field $\mathbb{R}$, where $F_3[x]$ is the set of all polynomials over $\mathbb{R}$ with degree less than or equal to 3. 5

(c) If $V(F)$ is a finite-dimensional vector space which is a direct sum of its two subspaces, U and W, then prove that

$$\dim V = \dim U + \dim W$$ 5

6. *(a)* If

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

then—

(i) find the linear transformation T corresponding to A with respect to the standard basis of $\mathbb{R}$;

(ii) evaluate rank T and nullity T. 3+4=7

(b) Find the characteristic polynomial of the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Also determine its eigenvalues. 3+2=5

(c) Show that the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (3x, 4x - y, 2x + 3y - z)$$

is invertible. 3

★★★