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(May/June)

MATHEMATICS

(Honours)

(Advanced Calculus—II)

(H–61)

Marks : 30

Time : 2 hours

The figures in the margin indicate full marks for the questions

Answer **two** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Define the terms 'limit point' and 'interior point' of a set. Find the limit points of the following sets : 1+1+2+2=6

    (i) $Q \cap [-1, 3]$, where Q is the set of rational numbers

    (ii) $\left\{ m + \frac{1}{n} : m, n \in \mathbb{N} \right\}$

   (b) State and prove Heine-Borel theorem. 6

   (c) Show that [0, 1] is a compact set. 3

2. *(a)* If $f:\mathbb{R}\to\mathbb{R}$ is continuous on a compact set S, then show that $f(S)$ is compact. 5

(b) If $A, B\subseteq\mathbb{R}$, then show that—

(i) $(A\cup B)'=A'\cup B'$

(ii) $\text{int}(A\cap B)=\text{int}(A)\cap\text{int}(B)$ 3+3=6

(c) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$ and let $f(x)=1\quad\forall\, x\in\mathbb{Q}$ show that

$$f(x)=1\ \ \forall\, x\in\mathbb{R}$$ 2

(d) Give an example of a function f from a bounded subset of $\mathbb{R}$ to $\mathbb{R}$ which does not attain its bounds. 2

UNIT—II

3. *(a)* Define directional derivative of a real-valued function f at a point (a, b). If all directional derivatives of a function $f:\mathbb{R}^2\to\mathbb{R}$ exist, does it imply that f is continuous at that point? Justify your answer. 1+4=5

(b) Show that the function $f(x)=x^2$ is not uniformly continuous on $\mathbb{R}$. 3

(c) If $f:[a, b]\to\mathbb{R}$ is continuous and strictly increasing, then show that f^{-1} is also continuous and strictly increasing. 4

(d) If $f(x, y)=\dfrac{xy}{x^2+y^2}$, do the partial derivatives and directional derivatives at $(0, 0)$ exist? Justify your answer. 3

4. *(a)* Let f be defined in a domain D of $\mathbb{R}^2$ and (a, b) be an interior point of D. Let—

(i) f_x exist at (a, b);

(ii) f_y be continuous at (a, b).

Show that f is differentiable at (a, b). 5

(b) Given $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x, y)=\frac{xy}{\sqrt{x^2+y^2}} \quad \text{if } (x, y)\neq(0, 0)$$

$$f(0, 0)=0$$

Show that f is continuous, possesses partial derivatives but is not differentiable at $(0, 0)$. 5

(c) If $f(x, y)=\left|x^2-y^2\right|\ \ \forall\ x, y,$ determine whether

$$f_{xy}(0, 0)=f_{yx}(0, 0)$$ 5

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