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(October)

MATHEMATICS

(Elective/Honours)

FIRST PAPER

(Algebra—I and Calculus—I)

Marks : 75

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

Answer **five** question, taking **one** from each Unit

UNIT—I

1. *(a)* If a finite set S has n elements, then prove that the power set of S has 2^n elements. 4

   (b) Show that the domain of definition of the function $f(x) = \log \dfrac{1-x}{1+x}$ is the interval $(-1, 1)$. Also, show that for $x_1, x_2 \in (-1, 1)$

   $$f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2} \right)$$ 2+3

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(c) Draw the graph of the function

$$f(x) = \begin{cases} 0 & , \quad x = 0 \\ 1-x, & 0 < x < 1 \\ 1 & , \quad x = 1 \end{cases}$$

The graph suggests that the function $f(x)$ assumes at least once every value between $f(0)$ and $f(1)$ and yet it has a discontinuity in $[0, 1]$. Justify this.   3+3

2. (a) Let $S = \{-2, -1, 0, 1, 2\}$. Give examples of the following :   2+2+2

   (i) A relation on S that is reflexive but not transitive

   (ii) A relation on S that is reflexive but not symmetric

   (iii) A relation on S that is symmetric but not reflexive

   (b) Using definition of limit, prove that

   $$\lim_{x \to 2} 5x = 10$$ 3

   (c) Discuss the continuity of the function $f(x) = \dfrac{|x|}{x}$ at $x = 0$ and $x = 1$.   3+3

UNIT—II

3. (a) Let $f : Q \to Q$ be defined by $f(x) = 2x + 3$, where $Q =$ set of rational numbers. Show that f is one-to-one and onto. Also, find a formula that defines the inverse function f^{-1}.   3+2

   (b) Give examples of—

   (i) matrices A, B such that $AB \neq BA$;

   (ii) matrices A, B such that $AB = 0$ but $A \neq 0$, $B \neq 0$.   2+2

   (c) Determine if the following system of equations is consistent and if so, find the solution :   2+4

   $$x - y + 2z = 4$$
   $$3x + y + 4z = 6$$
   $$x + y + z = 1$$

4. (a) If B is an idempotent matrix, show that $A = I - B$ is also idempotent and $AB = BA = 0$.   3+2

   (b) Consider the equation

   $$f(x) \equiv x^3 - 3x^2 + 4x - 3 = 0$$

   Find $f(A)$ and A^{-1} if

   $$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$ 4

(4)

(c) Applying elementary row operation, find the rank of the matrix

$$\begin{pmatrix} -4 & 1 & -1 & 2 \\ 1 & 0 & -1 & 0 \\ -5 & 2 & -5 & 4 \end{pmatrix}$$

6

UNIT—III

5. (a) Show that $f(x) = x^2$ for $-2 \leq x \leq 2$ is uniformly continuous. 4

(b) Find the slopes of the parabola $y = x^2$ at the vertex and at the point $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$.

Determine whether the tangent line to the parabola at the point $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$ makes an angle 45° with x-axis with justification. 3+2

(c) If $y = \cos(m\sin^{-1} x)$, show that—

(i) $(1 - x^2)y_2 - xy_1 + m^2 y = 0$;

(ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0.$ 2+4

6. (a) If the rate of change of y with respect to x is 5 and x is changing at 3 units per second, how fast is y changing? 3

(5)

(b) Find $\dfrac{d^2y}{dx^2}$, where $y = x\sqrt{x^2 + 9}$ at $x = 4$. 3

(c) Find the nth derivative of $\sqrt{x}$. 4

(d) Evaluate any two of the following : 2½×2=5

(i) $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$

(ii) $\lim\limits_{x \to 0} x\log x$

(iii) $\lim\limits_{x \to 0} x^{2 \sin x}$

UNIT—IV

7. (a) Integrate any two of the following : 4×2=8

(i) $\displaystyle\int \dfrac{x^2}{1 - x^4}\,dx$

(ii) $\displaystyle\int \log(1 + x)^{(1 + x)}\,dx$

(iii) $\displaystyle\int \dfrac{dx}{(1 - x)\sqrt{1 + x}}$

(b) Express

$$\int_a^b x^2 \, dx$$

as the limit of a sum and evaluate it. 4

(c) Use the properties of definite integral to show that

$$\int_0^{\pi/2} \log \tan x \, dx = 0$$

3

(6)

8. *(a)* Prove that

$$\int_{a-c}^{b-c} f(x+c)\,dx = \int_{a}^{b} f(x)\,dx \qquad 2$$

(b) Evaluate :

$$\underset{n \to \infty}{\text{Lt}} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2} \right] \qquad 4$$

(c) Obtain reduction formula for

$$\int_{0}^{\pi/2} \cos^n x\,dx$$

n being a positive integer greater than 1.
Hence, evaluate

$$\int_{0}^{1} x^2 \sqrt{1-x^2}\,dx \qquad 3+3$$

(d) Show that

$$\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2} \qquad 3$$

UNIT—V

9. *(a)* State the degree and order of the following differential equations :   1+1

    (i) $\dfrac{d^2 y}{dx^2} = x^3 \left(\dfrac{dy}{dx}\right)^2$

    (ii) $y\left(\dfrac{dy}{dx}\right)^3 + 2x\dfrac{dy}{dx} - y = 0$

(7)

(b) Obtain the differential equation of the system of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

     (λ is a variable parameter)

and hence, show that the system is self-orthogonal.   3+3

(c) Solve any *two* of the following :   3½×2=7

    (i) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

    (ii) $x\,dy - y\,dx + a(x^2+y^2)\,dx = 0$

    (iii) $\sec^2 y\dfrac{dy}{dx} + 2x\tan y = x^3$

10. *(a)* Find the general and singular solutions of $y = px + \sqrt{4p^2+1}$.   4

(b) Find $f(x)$, if $f'(x) = x\,f(x)$ and $f(0)=1$.   3

(c) Find the equation of the curve whose slope at any point (x, y) on it is xy and which passes through the point $(0, 1)$.   3

(d) Solve any *one* of the following :   5

    (i) $(D^2 - 9D + 20)y = x^2 e^{3x}$

    (ii) $(D^2 - D - 2)y = \sin 2x$

    (iii) $(D^2 - 2D + 1)y = 0; \quad y = 0, \quad Dy = 1$
       when $x = 0$

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