2019

(October)

MATHEMATICS

(Elective/Honours)

(GHS–11)

(Algebra–I & Calculus–I)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Prove that for any two sets A and B $(A \cup B)^c = A^c \cap B^c$, where A^c is the complement of A. 3

   (b) If $f(x+3) = 2x^2 - 3x + 1$, then find $f(x+1)$. 4

   (c) Find the domain and range of the function $f(x) = \dfrac{x^2}{x}$. Also draw the graph of $f(x)$. 2+3=5

(d) Give an example of a relation which is—

(i) reflexive, symmetric but not transitive;

(ii) symmetric and transitive but not reflexive;

(iii) reflexive and anti-symmetric. 1×3=3

2. (a) Using ε-δ definition of limit, show that

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$ 4

(b) For what value of a, $f(x) = 2ax + 3$, $x \neq 2$ and $f(2) = 3$ is continuous at $x = 2$? 3

(c) Let $\mathbb{Z}$ be the set of all integers and a relation R is defined as

$$R = \{(a, b) : a - b \text{ is even}\}$$

Is it an equivalence relation? Justify. 5

(d) In a group of 1000 people who can speak Khasi or Bengali; there are 750 who can speak Khasi and 400 who can speak Bengali. How many can speak Khasi only? How many can speak Bengali only? How many can speak both Khasi and Bengali? 3

UNIT—II

3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $x \in \mathbb{R}$. Examine if f is—

(i) injective;

(ii) surjective. $1\frac{1}{2}+1\frac{1}{2}=3$

(b) Give examples of—

(i) matrices A and B such that $AB \neq BA$;

(ii) matrices A and B such that $AB = 0$ but $A \neq 0$, $B \neq 0$. 2+2=4

(c) Solve the following system of linear equations with the help of Cramer's rule : 4

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$2x + 2y + 9z = 14$$

(d) Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix. 4

4. (a) If

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

then find AA' and $A'A$, where A' is the transpose of matrix A. 2+2=4

(b) Find the rank of the matrix

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

by reducing it to normal form. 6

(c) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

by elementary row operations.

UNIT—III

5. (a) Find the derivative of $\sin x$, $x > 0$ from the first principle. 4

(b) Find $\frac{dy}{dx}$ of the following (any *one*) : 4

(i) $x^y y^x = 1$

(ii) $y = \tan^{-1}\left(\frac{a+bx}{b-ax}\right)$

(c) If $\tan y = \frac{2t}{1-t^2}$ and $\sin x = \frac{2t}{1+t^2}$, then find $\frac{dy}{dx}$. 4

(d) Find the slope of the tangent line at the point $(0, 2)$ of the curve $8y = x^3 - 12x + 16$. 3

6. (a) State Leibnitz's theorem on the nth derivative of the product of two functions. Find y_n, if $y = \sqrt{x}$. 2+2=4

(b) If $y = (x^2 - 1)^n$, then prove that

$$(x^2-1)y_2 - (n-1)2xy_1 - 2ny = 0$$

and hence

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$ 5

(c) Use L' Hospital's rule to evaluate the following limits (any *one*) : 3

(i) $\underset{x\to\infty}{\text{Lt}} \frac{x^3}{e^x}$

(ii) $\underset{x\to 0}{\text{Lt}} \frac{\log(x^2)}{\log(\cot^2 x)}$

(d) Water is running into a conical reservoir, 10 cm deep and 5 cm radius at the rate 1.5 cc per minute. At what rate is the water level rising when the water is 4 cm deep? 3

UNIT—IV

7. (a) Evaluate any *two* of the following : 2×2=4

(i) $\int \frac{x}{\sqrt{x}+1} dx$

(ii) $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

(iii) $\int \frac{dx}{(\sin x + \cos x)^2}$

(b) Show that any *one* of the following : 3

(i) $\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{\pi}{2} \log \frac{1}{2}$

(ii) $\int_0^{\infty} \frac{dx}{(x+1)(x+2)} = \log 2$

(c) Let n be a positive integer and let $I_n = \int x^n e^{ax} dx$. Derive the reduction formula

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Hence find $\int x^5 e^{ax} dx$. 2+2=4

(d) Evaluate the following : 4

$$\underset{n \to \infty}{\text{Lt}} \left[\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right].$$

8. (a) Express $\int_0^1 (ax+b)dx$ as the limit of a sum and evaluate it. 4

(b) Using the properties of definite integral, show that 4

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$$

(Continued)

(c) Find $\int_0^3 f(x) dx$, where $f(x) = 2x$, when $0 \le x \le 2$ and $f(x) = x^2$, when $2 \le x \le 3$. 3

(d) Evalaute $\int_{-\infty}^{\infty} xe^{-x^2} dx$ if it converges. 4

UNIT—V

9. (a) Show that $Y = Ae^{2x} + Be^{-2x}$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4y = 0$$ 3

(b) Solve any *two* of the following : 2½×2=5

(i) $\frac{dy}{dx} = \frac{x^2 + x + 1}{y^2 + y + 1}$

(ii) $x + y\frac{dy}{dx} = 2y$

(iii) $(2x - y + 1)dx + (2y - x - 1)dy = 0$

(c) Solve : $\frac{dy}{dx} + \frac{y}{x} = y^2$ 3

(d) Find the equation of orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda^2} = 1$, where λ is a parameter. 4

(Turn Over)

10. *(a)* Solve any *two* of the following : 4×2=8

(i) $p^2 + p - 6 = 0$

(ii) $p^2 - (a+b)p + ab = 0$

(iii) $p^2 - p(e^x + e^{-x}) + 1 = 0$

$\left[p \text{ stands for } \dfrac{dy}{dx}\right]$

(b) Find the general and singular solution of $y = px + ap(1-p)$. 4

(c) Find the equation of the curve whose slope at any point (x, y) is xy and which passes through the point (0, 1). 3

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