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(February)

MATHEMATICS

(Elective / Honours)

(Algebra—II and Calculus—II)

(GHS–31)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Show that the set of all positive rational numbers forms an Abelian group under the composition defined by

$$a * b = \frac{ab}{2}$$ 5

(b) Show that the intersection of two subgroups of a group G is also a subgroup. 5

(c) If a and b are any two elements of a group G and H is any subgroup of G, then prove that

$$Ha = Hb \Rightarrow ab^{-1} \in H$$

where Ha = right coset of H in G generated by a. 5

2. *(a)* Let G be a group and a, b are elements of G. Prove that the equations $ax = b$ and $ya = b$, have unique solutions in G. 4

(b) Show that an infinite cyclic group has exactly two generators. 5

(c) Let G be a group and H a subgroup of G. Prove that G is the union of all left cosets of H in G and any two distinct left cosets of H in G are disjoint. 6

UNIT—II

3. *(a)* If a polynomial $f(x)$ is divided by $x - \alpha$, then show that the remainder is $f(\alpha)$. 3

(b) Find the value of k such that the polynomial $4x^3 - 3x^2 + 2x + k$ is divisible by $(x + 2)$. 2

(c) Solve the equation

$$x^4 - x^3 + 3x^2 + 31x + 26 = 0$$

if one root is $2 + 3i$. 5

(d) Prove that the equation

$$x^4 + 15x^2 + 7x - 11 = 0$$

has two real roots, one positive and the other negative and two other complex roots. 5

4. (a) If α, β, γ be the roots of the equation

$$x^3 + 2x^2 + 3x + 1 = 0$$

find the equation whose roots are

$$\alpha(\beta + \gamma),\ \ \beta(\gamma + \alpha),\ \ \gamma(\alpha + \beta)$$ 5

(b) Solve the equation $x^3 - 12x - 65 = 0$ by Cardan's method. 5

(c) If $2\cos\theta = a + \frac{1}{a}$, prove that

$$2\cos n\theta = a^n + \frac{1}{a^n}$$ 5

UNIT—III

5. (a) Prove that a convergent sequence is always bounded. Is the converse true? Justify your answer with example. 4+1=5

(b) Apply Cauchy's general principle of convergence to show that the sequence $\{u_n\}$ is convergent, if

$$u_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n-1}\frac{1}{n}$$ 5

(c) If

$$u_n = \frac{2.5.8\ldots(3n-1)}{3.7.11\ldots(4n-1)}$$

prove that $\{u_n\}$ is monotonic decreasing and its limit is zero. 5

6. (a) Test the convergence of any *two* of the following series : 3×2=6

(i) $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{1}{\sqrt{n(n+1)}}$

(ii) $\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$

(iii) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \ldots + \frac{2n-1}{n(n+1)(n+2)} + \ldots$

(b) Give an example of—

(i) an absolutely convergent series;

(ii) a conditionally convergent series with proper justification. 2+2=4

(c) Show that an absolutely convergent series is convergent. Determine the radius of convergence of the power series $1+2x+3x^2+4x^3+\ldots$. 3+2=5

UNIT—IV

7. *(a)* State and prove Lagrange's mean value theorem of differential calculus. 2+4=6

(b) Find the asymptotes of the curve

$$(x^2-y^2)(x+2y)+5(x^2+y^2)+x+y=0$$ 4

(c) Find the radius of curvature of the curve $y=4\sin x-\sin 2x$ at the point where $x=\frac{\pi}{2}$. 5

8. *(a)* Find the local maximum and local minimum values of the function

$$f(x)=\sin x+\frac{1}{2}\cos 2x, \quad 0\le x\le\frac{\pi}{2}$$ 5

(b) If $z=\log\left(\frac{x^4+y^4}{x+y}\right)$, then prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3$$ 5

(c) Let f be a function of two variables defined as follows :

$$f(x, y)=\begin{cases}\dfrac{(x^3+y^3)}{xy(x-y)} & \text{when} \quad x\ne y\\ 0 & \text{when} \quad x=y\end{cases}$$

Show that f is discontinuous at the origin. 5

UNIT—V

9. *(a)* Expand $\sin x$ in a finite series in powers of x, with remainder in Lagrange's form. 5

(b) Find the area included between the curve $y^2=9x$ and straight line $y=x$. 5

(c) Show that the length of the curve $y=\log\sec x$ between the points where $x=0$ and $x=\frac{1}{3}\pi$ is $\log(2+\sqrt{3})$. 5

10. *(a)* Find the volume of the paraboloid generated by the revolution about the x-axis of the parabola $y^2 = 4ax$ from $x = 0$ to $x = h$. 5

(b) Change the order of integration of

$$I = \int_0^a \int_{\frac{x}{a}}^{\sqrt{\left(\frac{x}{a}\right)}} (x^2 + y^2)\, dx\, dy$$ 5

(c) Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point $x = 2$. 5

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