2022

(February)

MATHEMATICS

(Honours)

(Elementary Number Theory and Advanced Algebra)

[GHS–51]

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

(Elementary Number Theory)

UNIT—I

1. *(a)* State whether the following are true or false with brief justification (any *five*) : 2×5=10

(i) If $a \mid b^2+1$, then $a \mid b^4+1$.

(ii) If $(a, b)=1$, then $(a+b, a-b)=1$ or 2.

(iii) n^2+n+41 is a prime for each positive integer n.

(iv) 13 does not divide $3n^2+3$ for each integer n.

(v) If $(a, b)=(c, d)$, then $[a, b]=[c, d]$.

(vi) $26 \mid \underline{|25} + 1$.

(b) Show that the product of four consecutive integers is divisible by 24. 2

(c) Find integers x and y such that $43x+64y=1$. 3

2. *(a)* Prove that there are infinitely many primes. 4

(b) Prove that a prime of the form $3k+1$ is necessarily of the form $6k+1$. 3

(c) Show that $7 \mid 3^{2n+1}+2^{n+2}$ for any positive integer n. 4

(d) Find the remainder when 25^{114} is divided by 48. 4

UNIT—II

3. *(a)* State Chinese remainder theorem. Solve the system of congruences

$$x \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 4$$
$$x \equiv 3 \pmod 5$$

1+4=5

(b) Prove that the fourth power of any integer must have one of the integers 0, 1, 5, 6 as its unit's digit. 3

(c) State and prove Fermat's theorem. 1+4=5

(d) Evaluate $\phi(3600)$ and $\sigma(240)$. 2

4. *(a)* If $\underline{|102}$ were written in ordinary decimal notation without factorial sign, then how many zeros would be there in a row at the right end? 3

(b) If n is an even positive integer, then show that

$$\sum_{d \mid n} \mu(d)\,\phi(d) = 0$$

4

(c) Prove that $\mu(n)$ is multiplicative and

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

4

(d) For what real values of x, is it true that

$$\left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] = [2x]\,?$$

4

(Advanced Algebra)

UNIT—III

5. *(a)* Define the terms normal subgroup and isomorphism between two groups. 1+1=2

(b) Show that

$$H = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1\right\}$$

is a normal subgroup of the group

$$G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc \neq 0\right\}$$

4

(c) Let $\theta : G \to G'$ be a group homomorphism. Show that θ is $1-1$ iff $\ker\theta = \{e\}$ where e is the identity of G. 3

(d) Let $\phi : G \to G'$ be a group homomorphism of G onto G' with kernel K. Show that

$$\frac{G}{K} \cong G'$$

6

6. *(a)* Prove that a finite integral domain is a field. 6

(b) Define characteristic of a ring. If D is an integral domain and if $na = 0$ for some $a \neq 0$ in D and some integer $n \neq 0$, then show that D is of finite characteristic. 1+4=5

(c) Show that $R = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$ is a commutative ring with identity element 1 under addition and multiplication of complex numbers. 4

UNIT—IV

7. *(a)* Show that any non-zero ideal of $\mathbb{Z}$ is of the form $m\mathbb{Z}$ for some positive integer m. 5

(b) Define a prime ideal of a ring. Prove that an ideal P of a ring R is prime iff R/P is an integral domain. 1+5=6

(c) Show that the only ideals of a field F are (0) and F itself. 4

8. *(a)* Let R be a PID. Show that if $p \in R$, $p \neq 0$, then Rp is a maximal ideal $\Leftrightarrow p$ is a prime. 5

(b) If R is a unique factorisation domain, then show that $R[x]$ is also a unique factorisation domain. 6

(c) Let D be an integral domain. Show that for $f(x),\ g(x) \in D[x]$

$$\deg\,(f(x) \cdot g(x)) = \deg\, f(x) + \deg\, g(x)$$ 4

UNIT—V

9. *(a)* For a vector space V, let $A(V)$ denote the set of all linear transformations of V to itself. Prove that $A(V)$ is a vector space. 3

(b) Define a subspace of a vector space. Prove that the intersection of two subspaces of a vector space V is a subspace of V. 2+4=6

(c) If T is an isomorphism of a vector space V onto a vector space W (V, W being vector spaces over a field F), then prove that T maps a basis of V to a basis of W. 6

10. *(a)* Define rank and nullity of a linear transformation. Show that if T is a linear transformation from V to V' (V, V' being vector spaces over a field F), then

$$\text{rank}\,(T) + \text{nullity}\,(T) = \dim V$$ 2+6=8

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x - y, 2y - z, z - x)$$

(i) Show that T is a linear transformation.

(ii) Find the matrix of T with respect to the standard basis of $\mathbb{R}^3$.

(iii) Find the rank and nullity of T.

2+3+2=7

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