Odd Semester, 2020

(Held in March, 2021)

MATHEMATICS

(Elective/Honours)

(GHS–11)

(Algebra–I & Calculus–I)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* For what values of x, the function $f(x)=\sqrt{(x-2)(x-3)}$ is not defined? 3

(b) Let $f(x)=\sin x$, $\phi(x)=\cos x$. Show that $\phi(2x)=1-2f^2(x)$. 2

(c) Let $A=\{x, y, z\}$, $B=\{y, w\}$. Determine

$$A\cup B,\ A\cap B,\ A\times B$$

and the power set $P(A)$. 4

(d) Examine if the following relations R on the set of integers are equivalence relations : 3×2=6

(i) xRy if and only if $x-y$ is an odd integer

(ii) xRy if and only if $x\le y$

2. *(a)* Let

$$f(x)=\frac{x-|x|}{x}\ ,\quad x\neq 0$$
$$=\quad 0\ ,\quad x=0$$

Examine if $f(x)$ is continuous at $x=0$. 4

(b) Establish the following limit using definitions : 3½

$$\lim_{x\to 3}\frac{1}{x}=\frac{1}{3}$$

(c) If $f(x)$ and $g(x)$ are continuous at $x=a$, prove that $f(x)+g(x)$ is also continuous at a. 3½

(d) Let

$$f(x)=4x+3\ ,\quad \text{when}\quad x\neq 4$$
$$=\quad 10\ ,\quad \text{when}\quad x=4$$

Obtain $\lim\limits_{x\to 4} f(x)$. 4

UNIT—II

3. *(a)* Let $f : X \to Y$ and $A \subseteq Y$, $B \subseteq Y$. Show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. 3

(b) If A and B are symmetric matrices of same order, show that AB is symmetric if and only if $AB = BA$. 4

(c) If A is an $n \times n$ matrix, prove that

$$|\operatorname{adj} A| = |A|^{n-1}$$ 3

(d) Using elementary row operations, compute the inverse of the matrix

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$ 5

4. *(a)* If A is an idempotent matrix, show that $I-A$ is also idempotent. 2

(b) Reduce the matrix

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

to the normal form. Hence find its rank. 7

(c) Examine if the following system of equations is consistent. If so, solve the system : 6

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

UNIT—III

5. *(a)* Define derivative of a function at an interior point of its domain. 2

(b) A function $f(x)$ is defined as

$$f(x) = 0 \quad , \quad 0 < x < 1$$
$$= 2 - x \quad , \quad 1 \le x \le 2$$

Show that $f'(1)$ does not exist. 3

(c) If

$$y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\cdots}}}}$$

prove that $\dfrac{dy}{dx} = \dfrac{\cos x}{2y - 1}$. 4

(d) Find $\dfrac{dy}{dx}$, if

$$y = \log(x + \sqrt{x^2 - a^2}) + \sec^{-1}\left(\frac{x}{a}\right)$$ 3

(e) Differentiate $\sin x$ with respect to x^2. 3

6. *(a)* If the area of a circle increases at a uniform rate, prove that the rate of the increase of the perimeter varies inversely as the radius. 3

(b) If $y = x^{n-1} \log x$, show that

$$y_n = \frac{(n-1)!}{x}$$ 4

(c) If $y = e^{1/x}$, find y_3. 3

(d) Evaluate (any *two*) : 2½×2=5

(i) $\underset{x\to\infty}{\mathrm{Lt}} (\sqrt{x^2+2x} - x)$

(ii) $\underset{x\to\frac{\pi}{2}}{\mathrm{Lt}} \dfrac{\tan 5x}{\tan x}$

(iii) $\underset{x\to 0}{\mathrm{Lt}} (\cos x)^{\cot^2 x}$

UNIT—IV

7. *(a)* Show that

$$\int \frac{x\,dx}{x^4 - x^2 - 2} = \frac{1}{6}\log\left|\frac{x^2-2}{x^2+1}\right| + c$$ 3

(b) Obtain a reduction formula for

$$\int_0^{\pi/2} \sin^m x \cos^n x dx$$

m, n being positive integers greater than 1. 5

(c) Evaluate : 4

$$\int_0^{\pi/2} \frac{\sin x\,dx}{\sin x + \cos x}$$

(d) Find the value of the improper integral

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}}$$

if it converges. 3

8. *(a)* Evaluate : 4

$$\lim_{n\to\infty}\left[\frac{1}{n+m} + \frac{1}{n+2m} + \cdots + \frac{1}{n+nm}\right]$$

(b) Using properties of definite integral, show that

$$\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log 2$$ 4

(c) Show that

$$\int_a^b \phi(x)\,dx = \int_0^{b-a} \phi(x+a)\,dx$$ 2

(d) Express the following integral as the limit of a sum and evaluate it : 5

$$\int_0^1 x^3\,dx$$

UNIT—V

9. *(a)* Eleminate a and b from the relation

$$xy = ae^x + be^{-x}$$ 3

(b) Solve (any *three*) : 3×3=9

(i) $e^{x-y}dx + e^{y-x}dy = 0$

(ii) $(x^2 + y^2)dy = xy\,dx$

(iii) $x\dfrac{dy}{dx} + y = y^2 \log x$

(iv) $\dfrac{dy}{dx} + xy = x$

(c) Find the orthogonal trajectories of $y^2 = 4ax$. 3

10. *(a)* Solve (any *two*) : 4×2=8

(i) $y = px + \dfrac{a}{p}$

(ii) $p^2 - 2xp + 1 = 0$

(iii) $p^2 + 2xp - 3x^2 = 0$

Here p stands for $\dfrac{dy}{dx}$

(b) Obtain the complete primitive and singular solution of

$$y = px + \sqrt{a^2p^2 + b^2}$$ 4

(c) Show that the equation of the curve whose slope at any point is $y + 2x$ and which passes through the origin is

$$y = 2(e^x - x - 1)$$ 3

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