Odd Semester, 2020

(Held in March, 2021)

MATHEMATICS

(Honours)

(GHS–52)

(Differential Equations and Advanced Dynamics)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Write Units I and II together in one answer script and Units III, IV and V together in another answer script

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* Solve :

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$$ 5

(b) Solve :

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$$ 5

(c) Verify that the equation

$$x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$$

is integrable and solve it. 5

2. *(a)* Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$$ 6

(b) Solve the simultaneous differential equations

$$\frac{dx}{dt} = 3x + 2y$$
$$\frac{dy}{dt} + 5x + 3y = 0$$ 6

(c) Solve : 3

$$(y + a)^2dx + zdy - (y + a)dz = 0$$

UNIT—II

$$\left(\text{In this unit, } p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}\right)$$

3. *(a)* Form a partial differential equation by eliminating the arbitrary function f from

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$ 6

(b) Solve : 5

$$x^2(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$$

(c) Find the complete integral of

$$p^3 + q^3 = 27z$$ 4

4. *(a)* Apply Charpit's method to find the complete integral of

$$2zx - px^2 - 2qxy + pq = 0$$ 5

(b) Find the complete integral and singular integral of

$$px + qy = pq$$ 6

(c) Solve : 4

$$q = xyp^2$$

UNIT—III

5. *(a)* A particle moving with a central acceleration μ / r^3 is projected from an apse at a distance a with velocity V. Show that the path is

$$r \cosh \frac{\sqrt{\mu - a^2V^2}}{aV}\theta = a$$

Or

$$r \cos \frac{\sqrt{a^2V^2 - \mu}}{aV}\theta = a$$

according as V is less than or greater than the velocity from infinity. 8

(b) A rough cycloid has its plane vertical and the vertex downwards. A heavy particle slides down the curve from rest at a cusp and comes to rest again at a point on the other side of the vertex where the tangent is inclined at an angle of $\pi / 4$ to the vertical. Show that

$$3\mu\pi + 4\log(1+\mu) = 2\log 2$$ 7

6. *(a)* A particle is describing an ellipse of eccentricity e about a centre of force at a focus. Prove with the usual notations that

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right);\ h^2 = \mu a(1 - e^2)$$

when the particle is at one end of a minor axis, its velocity is doubled. Prove that the new path is a hyperbola of eccentricity $\sqrt{9-8e^2}$. 3+5=8

(b) The base of a rough cycloid is horizontal and its vertex downwards. A bead slides along it starting from rest at the cusp and coming to rest at the vertex. Show that $\mu^2 e^{\mu\pi} = 1$, where μ is the coefficient of friction. 7

UNIT—IV

7. (a) The centre of mass of a plane lamina of mass M is G and GX, GY are principal axes of inertia at G. OX and OY are axes in the plane of the lamina parallel to GX and GY respectively. Prove that the product of inertia of the lamina with respect to OX and OY is $M\alpha\beta$ where α, β are the coordinates of G with respect to OX and OY. If the principal moments of inertia of the lamina are $Mn\alpha^2$ about GX and $Mn\beta^2$ about GY, and one of the principal axes of inertia at O is inclined at an angle of 45° to OX, prove that

$$\frac{n-1}{m-1} = \frac{\beta^2}{\alpha^2}$$

3+4=7

(b) For a coplanar rigid system, prove that the principal moments of inertia at a point are the extreme values of the moments of inertia at that point. Show further that these extreme values are given by

$$I_{\min} = \frac{1}{2}[A + B - \sqrt{(B-A)^2 + 4F^2}]$$

$$I_{\max} = \frac{1}{2}[A + B + \sqrt{(B-A)^2 + 4F^2}]$$

where A, B and F have their usual meanings. 8

8. (a) Define equimomental systems. State and prove necessary and sufficient conditions for two systems to be equimomental. 2+1+5=8

(b) A square of side a has particles of masses m, $2m$, $3m$ and $4m$ at its vertices. Show that the principal moments of inertia at the centre of the square are $2ma^2$, $3ma^2$, $5ma^2$ and find the directions of the principal axes. 3½+3½=7

UNIT—IV

9. (a) A circular hoop of radius a, rotating in a vertical plane with spin ω and with its centre at rest, is in contact with a rough plane inclined at angle α, the angle of friction for the surfaces in contact also being α. Show that, if the initial slip velocity is down the plane, the hoop remains stationary for a time $a\omega/(g\sin\alpha)$ and then the hoop rolls down the plane with acceleration

$$\frac{1}{2}g\sin\alpha$$ 8

(b) A uniform rod AB of mass M and length $2a$ lies at rest on a smooth horizontal table. An impulse J is applied at A in the plane of the table and perpendicular to the rod. Determine the velocity of the centroid and the angular velocity of the rod. 7

10. *(a)* A frame made of thin wire of line density ρ has the shape of a circle of radius a together with a diameter AB of the circle. The frame lies on a smooth horizontal table and is free to rotate about a fixed vertical axis through a point O on the circumference of the circle and on the diameter normal to AB. An insect of mass $\frac{1}{3}a\rho$ initially at rest at A begins to crawl along the diameter AB. If ϕ is the angle turned backwards by the frame in space after the insect has crawled a distance x along AB, show that

$$\{n^2a^2+(a-x)^2\}\dot{\phi}=a\dot{x}$$

where $n^2=12\pi+9$.

Prove also that when the insect reaches B, the frame will have rotated through an angle

$$\frac{1}{n}\tan^{-1}\left(\frac{2n}{n^2-1}\right)$$ 7+2=9

(b) A uniform rod is placed on a horizontal table with $\frac{2}{3}$rd of its length hanging over the edge. If the rod is at right angles to the edge when it is released, show that it will begin to slip when the rod has turned through an angle of $\tan^{-1}(\mu/2)$ where μ is the coefficient of friction between the rod and the table. 6

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