6/H–29 (vii) (Syllabus–2015)

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(July)

MATHEMATICS

(Honours)

(Advanced Calculus)

(GHS–61)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* If f is bounded and integrable in $[a, b]$, show that $|f|$ is integrable in $[a, b]$. Is the converse true? Justify your answer. 5+1=6

(b) A function f defined on the interval $[0, 1]$ as follows :

$$f(x) \quad \frac{1}{2^n}, \text{ when } \frac{1}{2^{n \;\; 1}} \quad x \quad \frac{1}{2^n} \quad n \quad 0, 1, 2, \ldots$$
$$0 \text{ , when } x \quad 0$$

Show that f is integrable on $[0, 1]$ and find the value of ${}_0^1 f(x)\,dx$. 5

(c) Show that every continuous function on a closed and bounded interval is Riemann integrable. 4

2. *(a)* Show that

$${}_0 \, x^{n \;\; 1} e^{\;\; x} dx$$

is convergent, if and only if, $n \quad 0$. 6

(b) Examine the convergence of the improper integral

$${}_0^2 \frac{dx}{x(2 \quad x)}$$ 4

(c) Show that

$${}_0 \frac{\sin x}{x} dx$$

is convergent. 5

UNIT—II

3. *(a)* Let f be continuous in $[a, b] \quad [c, d]$. Show that

$${}_c^d \; {}_a^b f(x, y) dx \; dy \qquad {}_a^b \; {}_c^d f(x, y) dy \; dx$$ 6

(b) If $|a| \quad 1$, show that

$${}_0 \log(1 \quad a\cos x)dx \qquad \log \frac{1}{2} \quad \frac{1}{2}\sqrt{1 \quad x^2}$$ 5

(c) Show that

$$_0 \frac{\tan \ ^1 ax}{x(1 \quad x^2)} dx \quad \frac{}{2} \log(1 \quad a)$$

if $a \quad 0$. 4

4. (a) Evaluate

$$f(\ , \ ) \quad _0 e \quad ^x \frac{\sin \ x}{x} dx$$

where $\quad 0$ and deduce that

$$_0 \frac{\sin \ x}{x} dx \quad \begin{matrix} \frac{}{2}, \text{ if} & 0 \\ 0, \text{ if} & 0 \\ \frac{}{2}, \text{ if} & 0 \end{matrix}$$

8

(b) Show that uniformly convergent improper integral of a continuous function is itself a continuous function. 4

(c) Show that $_0 e \quad ^{x^2} \cos yx \, dx$ is uniformly convergent in $] \quad , \quad [$. 3

UNIT—III

5. (a) Evaluate

$$_C (2x^2 \quad y^2) dx \quad (3y \quad 4x) dy$$

where C is the triangle ABC whose vertices are

$$A \quad (0, 0), \ B \quad (2, 0), \ C \quad (2, 1)$$ 5

(b) Evaluate $\quad xy(x \quad y) dx \, dy$ over the area between the curves $y \quad x^2$ and $y \quad x$. 5

(c) Change the order of integration in the double integral

$$I \quad _0^{2a} \ _{\frac{x^2}{4a}}^{3a \quad x} \quad (x, \ y) dx \, dy$$

where $\quad (x, y)$ is a continuous function. 5

6. (a) State Green's theorem in $\mathbb{R}^2$. Use it to evaluate the integral

$$_C (x^2 \quad xy^3) dx \quad (y^2 \quad 2xy) dy$$

where C is the square with vertices (0, 0), (2, 0), (2, 2), (0, 2). 5

(b) Changing variables from Cartesian to polar coordinates, evaluate the integral

$$_E \sqrt{x^2 \quad y^2} \, dx \, dy$$

where E is the region bounded by the circles $x^2 \quad y^2 \quad 4$ and $x^2 \quad y^2 \quad 9$. 5

(c) Apply Gauss' divergence theorem to evaluate

$$_S \ x^2 dy \, dz \quad y^2 dz \, dx \quad 2z(xy \quad x \quad y) dx \, dy$$

where S is the surface of the cube $0 \quad x \quad 1, 0 \quad y \quad 1, 0 \quad z \quad 1$. 5

UNIT—IV

7. *(a)* Determine whether the following statements are *True* or *False* with brief justification : 6

(i) The intersection of an arbitrary collection of closed sets is closed.

(ii) The derived set of every set is a closed set.

(iii) The set $\mathbb{Q}$ of all rational numbers is a closed set.

(b) State and prove Bolzano-Weierstass theorem in $\mathbb{R}$. 5+1=6

(c) Let A and B be subsets of $\mathbb{R}$. Then show that $\overline{A \quad B} \quad \bar{A} \quad \bar{B}$. 3

8. *(a)* Let A be an infinite subset of a compact set Y. Then prove that A has a limit point in Y. 5

(b) Show that the function given by

$$f(x, y) \qquad \begin{matrix} xy\dfrac{x^2 \quad y^2}{x^2 \quad y^2}, & \text{if } (x, y) \quad (0, 0) \\ 0 & , \text{ if } (x, y) \quad (0, 0) \end{matrix}$$

is continuous at $(0, 0)$. 5

(c) Show that a continuous and strictly monotonic increasing function f on $[a, b]$ admits an inverse which is also continuous. 5

UNIT—V

9. *(a)* Let f be a real-valued continuous function with compact domain $D \quad \mathbb{R}^n$. Show that f is uniformly continuous in D. 5

(b) Show that the function $f(x, y) \quad \sqrt{|xy|}$ is not differentiable at the origin, but the partial derivatives exist at $(0, 0)$. 5

(c) Show that the function

$$f(x, y) \qquad \begin{matrix} xy\dfrac{(x^2 \quad y^2)}{(x^2 \quad y^2)}, & (x, y) \quad (0, 0) \\ 0 & , \text{ if } (x, y) \quad (0, 0) \end{matrix}$$

does not satisfy conditions of Schwarz's theorem. 5

10. *(a)* Let f be a real-valued function defined in a domain $D \quad \mathbb{R}^n$ and let a be an interior point of D. If f is differentiable at a, then show that f admits partial derivatives at a. 5

(b) If

$$u \quad \frac{y^2}{2x}, \; v \quad \frac{x^2 \quad y^2}{2x}$$

then show that $\dfrac{(u,\ v)}{(x,\ y)} \quad \dfrac{y}{2x}$. 5

(c) Find the directional derivatives of the function $f : \mathbb{R}^2 \quad \mathbb{R}$ defined by

$$f(x,\ y) \quad \sqrt{x^2 \quad y^2}$$

at the point $(a_1,\ a_2)$ in the direction $(x,\ y)$. 5

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