2/EH–29 (ii) (Syllabus–2019)

2022

(May/June)

MATHEMATICS

(Elective/Honours)

(Geometry and Vector Calculus)

(GHS–21)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* Find the angle through which a set of rectangular axes must be turned without the change of origin so that the expression $7x^2+4xy+3y^2$ will be transformed into the form $a'x^2+b'y^2$. 5

(b) For what values of k will the equation

$$3x^2+kxy-3y^2+29x-3y+18=0$$

represent a pair of straight lines? 5

(c) Reduce the equation

$$9x^2+24xy+16y^2-126x+82y-59=0$$

to standard form. 5

2. (a) Prove that the equation of the polar of origin with respect to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is $gx+fy+c=0$. 6

(b) Prove that the two hyperbolas

$$3x^2-4xy+3x+4=0$$

and

$$2x^2+3xy-2x+5=0$$

have a common asymptote and find the other asymptote. 4

(c) Find the equation of the tangent to the conic $4x^2+3xy+2y^2-3x+5y+7=0$ at the point (1, – 2). 5

UNIT—II

3. (a) Prove that the sum of the reciprocals of two perpendicular focal chords of a conic is constant. 5

(b) Show that the locus of the poles of normal chords of the parabola $y^2=4ax$ is $(x+2a)y^2+4a^3=0$. 5

(c) A double ordinate of the parabola $y^2=4px$ is of length $8p$. Prove that the lines from the vertex to its two ends are at right angles. 5

4. (a) If e and e' are the eccentricities of a hyperbola and its conjugate, then show that

$$\frac{1}{e^2}+\frac{1}{e'^2}=1$$ 5

(b) Find the equations of the common tangents to the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$ 5

(c) For the ellipse $8x^2+12y^2=96$, find a pair of semi-conjugate diameters inclined at an angle $\tan^{-1}(7)$. 5

UNIT—III

5. (a) Find the equation of the plane through (2, – 3, 1) normal to the line joining the points (3, 4, – 1) and (3, – 1, 5). 5

(b) Find the equation to the locus of the point whose distance from the origin is equal to its perpendicular distance from the plane $2x+3y-6z=9$. 5

(c) Find the equation of the sphere through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touching the plane $z=0$. 5

6. (a) Find the equations of the tangent planes to the sphere $x^2+y^2+z^2=49$ which pass through the lines

$$2x+z-21=0=3y-z+14$$ 5

(b) Find the equation of the cylinder generated by the lines parallel to the line

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$$

and intersecting the guiding curve $z=3$, $x^2+y^2=4$. 5

(c) Find the enveloping cone of the sphere $x^2+y^2+z^2-2x+4z=1$ with its vertex at (1, 1, 1). 5

UNIT—IV

7. (a) Given two vectors $\vec{a}=\hat{i}+\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$, find a unit vector $\vec{c}$ perpendicular to the vector $\vec{a}$ and coplanar with $\vec{a}$ and $\vec{b}$. Find also a vector $\vec{d}$ perpendicular to both $\vec{a}$ and $\vec{c}$. 5

(b) Prove that

$$[\vec{p}\ \vec{q}\ \vec{r}][\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix}\vec{p}\cdot\vec{a} & \vec{p}\cdot\vec{b} & \vec{p}\cdot\vec{c}\\ \vec{q}\cdot\vec{a} & \vec{q}\cdot\vec{b} & \vec{q}\cdot\vec{c}\\ \vec{r}\cdot\vec{a} & \vec{r}\cdot\vec{b} & \vec{r}\cdot\vec{c}\end{vmatrix}$$

Hence deduce that

$$[\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}$$ 5

(c) Reduce the expression

$$(\vec{b}+\vec{c})\cdot\{(\vec{c}+\vec{a})\times(\vec{a}+\vec{b})\}$$

to its simplest form and prove that it vanishes when $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar. 5

8. (a) If the volume of the parallelopiped whose edges are represented by $-12\hat{i}+\lambda\hat{k}$, $3\hat{j}-\hat{k}$, $2\hat{i}+\hat{j}-5\hat{k}$ is 546, then find the value of λ. 5

(b) Prove that if $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, then $\vec{a}$ and $\vec{c}$ are parallel. Is the converse true? 5

(c) Show that the necessary and sufficient condition that a proper vector $\vec{u}$ which is a function of a scalar variable t, always remains parallel to a fixed line is that

$$\vec{u} \times \frac{d\vec{u}}{dt} = 0$$

5

UNIT—V

9. (a) Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at any time t. 5

(b) Find the maximum value of the directional derivative of $\phi = x^2 + z^2 - y^2$ at the point (1, 3, 2). Find also the direction in which it occurs. 5

(c) Find the directional derivative of $f = xy + yz + zx$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$ at (1, 2, 0). 5

10. (a) Find the constants a, b, c so that the vector

$$\vec{w} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

becomes irrotational. 5

(b) If $u = 3x^2y$, $v = xz^2 - 2y$, then prove that

$$\text{grad}(\text{grad}\, u \cdot \text{grad}\, v) = (6yz^2 - 12x)\hat{i} + 6xz^2\hat{j} + 12xyz\hat{k}$$ 5

(c) Show that $\text{div}(\text{grad}\, r^n) = n(n+1)r^{n-2}$, where $r = |\vec{r}|$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. 5

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