6/H–29 (x) (Syllabus–2019)

2022

(May/June)

MATHEMATICS

(Honours)

(Advanced Algebra)

(H–62)

Marks : 45

Time : 2 hours

The figures in the margin indicate full marks for the questions

Answer **three** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Let H be a non-empty subset of a group G. Prove that H is a subgroup of G if and only if $\forall a, b \in H$ $ab \in H$ and $\forall a \in H$, $a^{-1} \in H$. Also prove that H is a normal subgroup of G if and only if $gHg^{-1} = H$, $\forall g \in G$. 5+4=9

(b) Prove the following assertions : 3+3=6

(i) If H is a subgroup of a group G such that $x^2 \in H$ for all $x \in G$, then H is a normal subgroup of G.

(ii) If N and M are normal subgroups of a group G such that $N \cap M = \{e\}$, where e is the identity element of G, then $nm = mn$ for all $n \in N$ and for all $m \in M$.

2. (a) Show that the characteristic of an integral domain is either zero or a prime number. 5

(b) Show that every field is an integral domain. 4

(c) Are the following statements true or false? Answer with brief justification : 3×2=6

(i) If A and B are ideals of a ring such that $A \cap B = \{0\}$, then $\forall a \in A$, $\forall b \in B$; $ab = 0$.

(ii) Every prime ideal of an integral domain is necessarily a maximal ideal.

UNIT—II

3. (a) Show that if K is a field and R is a ring, then any non-zero homomorphism $f : k \to R$ is necessarily one-one. 5

(b) Define the terms prime and irreducible elements of an integral domain. 2

(c) If $f : R \to S$ is a ring homomorphism and I is an ideal of R, then is it necessarily true that $f(I)$ is an ideal of S? Answer with justification. 3

(d) Prove that in a principal ideal domain, every irreducible element is a prime element. 5

4. (a) Let K be a field, R a ring and $f : R \to K$ be an onto ring homomorphism. Show that $\ker(f)$ is a maximal ideal of R. 5

(b) Is the intersection of two maximal ideals of a principal ideal domain R again a maximal ideal of R? Justify your answer. 5

(c) Show that a maximal ideal of a ring is a prime ideal. Is the converse true? Justify your answer. 4+1=5

UNIT—III

5. (a) Let V be a vector space over a field F and let V_1, V_2 be subspaces of V. Then show that

$$W = V_1 + V_2 = \{v = v_1 + v_2 : v_1 \in V_1,\ v_2 \in V_2\}$$

is a subspace of V. 5

(b) Determine the values of a in $\mathbb{R}$ for which the vectors $(0, 1, a)$, $(1, a, 1)$, $(a, 1, 0)$ are linearly dependent in $\mathbb{R}^3$. 4

(c) Find the characteristic roots and characteristic vectors of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$ 6

6. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(x, y, z) = (x + y, y + z, 5z), \forall (x, y, z) \in \mathbb{R}^3$$

Find the matrix of T w.r.t. the bases $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ in $\mathbb{R}^3$. 5

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(x, y, z) = (x, x + y, x + y + z)$$

Show that f is a linear transformation. Also find image of f and kernel of f. 7

(c) Define the terms (i) minimum polynomial, (ii) characteristic root and (iii) characteristic vector of a linear transformation. 3

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