2022

(May/June)

MATHEMATICS

(Honours)

(**Elementary Differential Geometry**)

(HOP–7)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. (a) Find the length of the circular helix

$$\mathbf{r}(u) = (a\cos u)\mathbf{i} + (a\sin u)\mathbf{j} + (cu)\mathbf{k},$$

$-\infty < u < \infty$ from $(a, 0, 0)$ to $(a, 0, 2\pi c)$. 5

(b) Find the equation of the tangent line at the point u on the circular helix

$$x = a\cos u,\ y = a\sin u,\ z = cu$$ 5

(c) Show that if a curve is given in terms of a general parameter u, then equation of the osculating plane is

$$[\mathbf{R} - \mathbf{r},\ \dot{\mathbf{r}},\ \ddot{\mathbf{r}}] = 0$$ 5

2. (a) For the curve $x = 3t,\ y = 3t^2,\ z = 2t^3$, show that any plane meets it in three points and deduce the equation to the osculating plane at $t = t_1$. 5

(b) Define the terms 'principal normal' and 'binormal'. Deduce the equations of principal normal and binormal. 2+2+6=10

UNIT—II

3. (a) Show that a necessary and sufficient condition that a curve be a straight line is that $\kappa = 0$ at all points. 5

(b) If the tangent and the binormal at a point of a curve make angles θ, ϕ respectively with a fixed direction, show that

$$\frac{\sin\theta\, d\theta}{\sin\phi\, d\phi} = -\frac{\kappa}{\tau}$$

where κ and τ have their usual meanings. 6

(c) Define the curvature κ and torsion τ of a twisted curve. 2+2=4

4. (a) Prove Serret-Frenet formulas

$$\frac{d\hat{t}}{ds} = \kappa\hat{n},\ \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t},\ \frac{d\hat{b}}{ds} = -\tau\hat{n}$$ 8

(b) Calculate the curvature and torsion of the cubic curve given by

$$\mathbf{r} = (u,\ u^2,\ u^3)$$ 7

UNIT—III

5. (a) Prove that the angle between the principal normals at two consecutive points O and P on a curve is

$$s\sqrt{(\rho^{-2} + \sigma^{-2})}$$

where s is the length of the arc OP. 5

(b) Define the spherical indicatrix of the tangent. Also deduce the curvature and torsion of the spherical indicatrix of the tangents. 1+4+5=10

6. (a) A curve is drawn on a sphere of radius a, and the principal normal at a point P makes angle θ with the radius of the sphere at P. Prove that

$$\rho = a\cos\theta,\ \frac{1}{\sigma} = \pm\frac{d\theta}{ds}$$ 7

(b) Prove that the normal plane of any point to the locus of the centre of circular curvature of any curve bisects the radius of spherical curvature at the corresponding point of the given curve. 8

UNIT—IV

7\. *(a)* Find the equations of the tangent plane and normal to the surface $xyz = 4$ at the point (1, 2, 2). 3+3=6

(b) Find a unit normal vector to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$. 5

(c) Calculate the fundamental magnitudes for the conicoid $x = u\cos v$, $y = u\sin v$, $z = f(v)$ where u, v are parameters. 4

8\. *(a)* Show that if L, M, N vanish everywhere on a surface, then the surface is a part of a plane. 5

(b) Deduce the formulae

$$H\mathbf{N} \times \mathbf{N}_1 = M\mathbf{r}_1 - L\mathbf{r}_2 \text{ and}$$
$$H\mathbf{N} \times \mathbf{N}_2 = N\mathbf{r}_1 - M\mathbf{r}_2$$ 3+3=6

(c) Find the value of *(i)* first curvature, *(ii)* Gaussian curvature at any point of the right helicoid $\mathbf{r} = (u\cos v, u\sin v, av)$. 4

UNIT—V

9\. *(a)* Find the asymptotic lines of the paraboloid of revolution $z = x^2 + y^2$. 5

(b) If the first, second and third fundamental forms are denoted by *I*, *II*, *III* respectively, prove that $KI - 2\mu II + III = 0$ where K is the Gaussian curvature and μ is the mean curvature. 7

(c) Is the surface $e^z \cos x = \cos y$, a minimal surface? Justify your answer. 3

10\. *(a)* Prove that the necessary and sufficient conditions that the parametric curves on a surface form an isometric system, are

$$F = 0, \quad \frac{\partial^2}{\partial u \partial v} \log\left(\frac{E}{G}\right) = 0$$ 8

(b) Prove that the lines of curvature on a surface of revolution form an isothermal system. 7

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