6/H–29 (viii) (b) (Syllabus–2015)

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(April)

MATHEMATICS

(Honours)

(Operation Research)

(HOPT-62 : OP2)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. (a) A cold drinks company has two bottling plants, located at two different places. Each plant produces three different drinks A, B and C. The capacity of the two plants, in number of bottles per day, are as follows :

	Product A	*Product B*	*Product C*
Plant I	1500	3000	2000
Plant II	1500	1000	5000

A market survey indicates that during any particular month there will be a demand of 20000 bottles of A, 40000 bottles of B and 44000 bottles of C. The operating costs, per day, of running plant I and plant II are respectively 600 and 400 monetary units. How many days should the company run each plant during the month so that the production cost is minimized while still meeting the market demand? 10

(b) Solve the following LPP graphically : 5

Maximize $Z = x_1 + x_2$

subject to the constraints

$$x_1 + x_2 \le 1$$
$$-3x_1 + x_2 \ge 3$$
$$x_1 \ge 0,\ x_2 \ge 0$$

2. *(a)* Old hens can be bought for ₹ 2 each but young one costs ₹ 5 each. The old hens lay 3 eggs per week and young ones 5 eggs per week, each egg being worth 30 paise. A hen costs ₹ 1 per week to feed. There are only ₹ 80 to be spent on purchasing the hens and at the most 20 hens can be accommodated in the space. Formulate this problem as an LPP to determine each kind of hen that should be bought to have the maximum profit per week. 8

(Continued)

(b) Prove that the set of all feasible solutions to a linear programming problem is a closed convex set. 7

UNIT—II

3. *(a)* Explain the following terms : 2½+2½=5

(i) Basic feasible solution of an LPP

(ii) Unrestricted variable

(b) Rewrite in standard form the following LPP : 5

Minimize $Z = 2x_1 + x_2 + 4x_3$

subject to the constraints

$$-2x_1 + 4x_2 \le 4$$
$$x_1 + 2x_2 + x_3 \ge 5$$
$$2x_1 + 3x_3 \le 2$$
$$x_1 \ge 0,\ x_2 \ge 0$$

and x_3 unrestricted in sign.

(c) Write the dual problem of the following : 5

Minimize $Z = 4x_1 + 6x_2 + 18x_3$

subject to the constraints

$$x_1 + 3x_2 \ge 3$$
$$x_2 + 2x_3 \ge 5$$

and $x_j \ge 0 \quad (j = 1, 2, 3)$.

(Turn Over)

4. (a) Explain the simplex procedure to solve a linear programming problem. 9

(b) Construct the initial simplex table for the following LPP : 6

$$\text{Maximize } Z = 3x_1 - x_2$$

subject to

$$2x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 3$$
$$x_2 \leq 4$$
$$x_1,\ x_2 \geq 0$$

UNIT—III

5. (a) Use simplex method to solve the following LPP : 10

$$\text{Maximize } Z = 2x_1 + 3x_2$$

subject to the constraints

$$x_1 + x_2 \leq 4$$
$$-x_1 + x_2 \leq 1$$
$$x_1 + 2x_2 \leq 5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

(b) Describe briefly the following : 3+2=5

(i) Maximin-Minimax principle

(ii) Two-person zero-sum game

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6. (a) For the game with the following pay-off matrix

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} -4 & -2 & -2 & 3 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ -6 & -5 & -2 & -4 & 4 \\ 3 & 1 & -6 & 0 & -8 \end{bmatrix} \end{array}$$

determine the best strategies for player A and player B, and also the value of the game for them. Is the game (i) strictly determinable and (ii) fair? 9

(b) Consider the game G with the following pay-off matrix :

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 2 & 6 \\ -2 & \lambda \end{bmatrix} \end{array}$$

(i) Show that G is strictly determinable, whatever λ may be.

(ii) Determine the value of G.

(iii) What is the optimal strategy for player A and player B? 6

UNIT—IV

7. (a) A company is currently involved in negotiations with its union on the upcoming wage contract. Positive signs in the pay-off represent wage increase

(Turn Over)

while negative signs represent wage reduction. What are the optimal strategies for the company as well as the union? What is the game value? 10

Pay-off

$$\begin{array}{c c} & \textit{Company strategies} \\ \textit{Union strategies} & \begin{bmatrix} 0{\cdot}25 & 0{\cdot}20 & 0{\cdot}14 & 0{\cdot}30 \\ 0{\cdot}27 & 0{\cdot}16 & 0{\cdot}12 & 0{\cdot}14 \\ 0{\cdot}35 & 0{\cdot}08 & 0{\cdot}15 & 0{\cdot}19 \\ -0{\cdot}02 & 0{\cdot}08 & 0{\cdot}13 & 0{\cdot}00 \end{bmatrix} \end{array}$$

(b) Solve the following game : 5

$$\begin{array}{c c} & \textit{Player } B \\ \textit{Player } A & \begin{bmatrix} 1 & 3 & 2 \\ 7 & -5 & 1 \\ 4 & -1 & 2 \end{bmatrix} \end{array}$$

8. (a) Solve the following 3×2 game graphically : 5

$$\begin{array}{c c} & \textit{Player } B \\ \textit{Player } A & \begin{bmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{bmatrix} \end{array}$$

(b) Solve the following game by linear programming technique : 10

$$\begin{array}{c c} & \textit{Player } B \\ \textit{Player } A & \begin{bmatrix} 4 & 1 & -3 \\ 3 & 1 & 6 \\ -3 & 4 & -2 \end{bmatrix} \end{array}$$

(Continued)

UNIT—V

9. (a) Describe briefly the following : 2½+2½=5

(i) State transition matrix

(ii) Absorbing states

(b) There are three grocery stores A, B, C in a town. The stores conducted a study on the number of customers that each store retained, gained or lost from time T_1 to T_2 assuming that the number of total customers during the period remained same. The following is the gain and loss results of the study :

Stores	Total customers at time T_1	Total customers gain from			Total customers loss to			Total customers at time T_2
		A	B	C	A	B	C	
A	500	0	100	0	0	50	50	500
B	400	50	0	25	100	0	0	375
C	500	50	0	0	0	25	0	525

Construct the state transition matrix. 10

10. (a) State whether the following are True or False and give a brief justification : 1+1=2

(i) $v = \left(\frac{1}{3}, 0, \frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$ is a probability vector.

(Turn Over)

(ii) $A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ is a stochastic matrix.

(b) Show that

$$(cf + ce + de,\ af + bf + ae,\ ad + bd + bc)$$

is a fixed point of the matrix

$$P = \begin{bmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{bmatrix}$$ 3

(c) A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is 0·6. However, if he loses a game, the probability of losing the next game is 0·7. There is an even chance that the gambler wins the first game.

(i) What is the probability that he wins the second game?

(ii) What is the probability that he wins the third game?

(iii) In the long run, how often will he win? 10

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2019

(April)

MATHEMATICS

(Honours)

(Advanced Calculus)

(GHS-61)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* Prove that a monotonic function f on $[a, b]$ is integrable on $[a, b]$. 5

(b) A function f is defined on $[0, 1]$ as follows :

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q}, \text{ a non-zero rational number} \\ 0, & \text{if } x = 0 \text{ or } x \text{ is irrational} \end{cases}$$

Show that f is integrable on $[0, 1]$ and that $\int_0^1 f = 0$. 5

(Turn Over)

(c) If f is positive and monotonically decreasing on $[1, \infty[$, show that the sequence $\{A_n\}$, where

$$A_n = \{f(1) + f(2) + \cdots + f(n)\}$$

is convergent. 5

2. (a) Determine whether the following improper integral is convergent or not : 5

$$\int_0^1 \frac{dx}{x^{1/3}(1-x)^{1/2}}$$

(b) If ϕ is continuous on $[0, \infty[$ and $\lim_{x\to 0} \phi(x) = \phi_0$, $\lim_{x\to\infty} \phi(x) = \phi_1$, show that

$$\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x} dx = (\phi_0 - \phi_1)\log\frac{b}{a}$$ 6

(c) Show that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x} dx = \log\frac{b}{a}$$ 4

UNIT—II

3. (a) Let f and f_y be continuous on $[a, b] \times [c, d]$. Let g_1 and g_2 be two derivable functions on $[c, d]$ such that $g_1(y), \; g_2(y) \in [a, b] \quad \forall\, y \in [c, d]$. Show that the function ϕ, where

$$\phi(y) = \int_{g_1(y)}^{g_2(y)} f(x, y)\,dx$$

is differentiable on $[c, d]$ and that

$$\phi'(y) = \int_{g_1(y)}^{g_2(y)} f_y(x, y)\,dx + g_2'(y) f(g_2(y), y) - g_1'(y) f(g_1(y), y)$$

6

(Continued)

(b) Prove that

$$\int_0^a \frac{\log(1+ax)}{1+x^2} dx = \frac{1}{2}\log(1+a^2)\tan^{-1} a$$ 5

(c) By considering the identity

$$\frac{1}{y} = \int_0^\infty e^{-xy}\,dx$$

show that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log\frac{b}{a}$$ 4

4. (a) If f is continuous on $[a, \infty[\times [c, d]$ and the integral

$$\phi(y) = \int_a^\infty f(x, y)\,dx$$

is uniformly convergent, show that ϕ can be integrated under the integral sign. 5

(b) Show that $\int_0^\infty \frac{\cos xy}{\sqrt{1-x^2}} dx$ is uniformly convergent on $\mathbb{R}$. 3

(c) By differentiating under the integral sign show that

$$\int_0^\infty e^{-x^2 - \frac{a^2}{x^2}} dx = \frac{1}{2}\sqrt{\pi}e^{-2|a|}$$ 7

(Turn Over)

UNIT—III

5. (a) Show that

$$\int_C (x-y)^3 dx + (x-y)^3 dy = 3\pi a^4$$

where C is the circle $x^2+y^2=a^2$ in the anticlockwise direction. 6

(b) By changing the order of integration in

$$\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx dy$$

show that

$$\int_0^\infty \frac{\sin rx}{x} dx = \frac{\pi}{2}$$

where $r>0$. 5

(c) Change the order of integration in

$$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y) \, dx dy$$ 4

6. (a) Express as a repeated integral the double integral

$$\iint f(x, y) \, dx dy$$

taken over the interior of the quadrilateral

$x+y=0,\ x-y=0,\ 2x-y=1,\ 2x-3y+5=0$ 5

(b) State Gauss theorem and use it to evaluate

$$\iint_S [(x^3-yz) dydz - 2x^2 y \, dzdx + 2 dxdy]$$

taken over the surface of the cube bounded by the planes

$x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a$ 1+4=5

(c) Evaluate the double integral

$$\iint \sqrt{x(2a-x)+y(2b-y)} \, dxdy$$

over the interior of the circle

$x^2+y^2-2ax-2by=0$ 5

UNIT—IV

7. (a) Determine whether the following statements are True or False with brief justification : 2×3=6

(i) The set $Q \cap [a, b]$ is closed in $\mathbb{R}$.

(ii) The set $\{0\} \cup \left\{1-\frac{1}{n} : n \in \mathbb{N}\right\}$ has two limit points.

(iii) Every infinite subset of $[3, \infty[$ has a limit point.

(b) State and prove Heine-Borel theorem. 6

(c) Show that $(A \cap B)' \subseteq A' \cap B'$ for any two sets A, B in $\mathbb{R}^n$ (where X' is the set of limit points of X). 3

8. (a) Show that a function $f : X \to Y$, where $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ is continuous on X if and only if $f^{-1}(V)$ is open for every open set V in Y. 6

(b) If f is continuous on $\mathbb{R}$ and $f(x) = 2$ when x is rational, show that $f(x) = 2 \;\; \forall x$. 2

(c) Show that

$$\lim_{m \to \infty} \lim_{n \to \infty} \cos^{2n}(m!\pi x) = \begin{cases} 0, & \text{if } x \notin Q \\ 1, & \text{if } x \in Q \end{cases}$$ 3

(d) Justify whether the following are True or False : 2×2=4

(i) Let F be the collection of sets in $\mathbb{R}^n$ and $S = \bigcup_{A \in F} A$. If x is a limit point of S, then x is a limit point of at least one A in F.

(ii) If S is closed and T is compact in $\mathbb{R}^n$, then $S \cap T$ is compact.

UNIT—V

9. (a) Define uniform continuity of a function and show that $f(x) = 3x^2 + 2x + 1$ is not uniformly continuous on $[0, \infty[$. 3

(b) For a function f defined on $[0, 1]$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 1 - x, & \text{if } 1 - x \notin \mathbb{Q} \end{cases}$$

show that (i) f is continuous only at $x = \frac{1}{2}$ and (ii) f takes every value between 0 and 1. 4

(c) State and prove intermediate value theorem. 4

(d) Let f be strictly increasing on a set S in $\mathbb{R}$. Show that f^{-1} exists and is strictly increasing. 4

10. (a) Give definitions of the following : 1+1=2

(i) Partial derivative of a function

(ii) Differentiable function at a point $(a, b) \in \mathbb{R}^2$

(b) Let f be defined in a domain $D \subseteq \mathbb{R}^2$. Let (a, b) be an interior point of D and let (i) f_x exist at (a, b) and (ii) f_y be continuous at (a, b). Show that f is differentiable at (a, b). 5

(c) If

$$f(x, y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & \text{if } xy \neq 0 \\ x^2 \sin\frac{1}{x} & , \text{if } x \neq 0 \\ y^2 \sin\frac{1}{y} & , \text{if } y \neq 0 \\ 0 & , \text{if } x = 0 = y \end{cases}$$

show that f is differentiable at (0, 0) but f_x and f_y are not continuous at (0, 0). 5

(d) Show that a function which is differentiable at a point $(x, y) \in \mathbb{R}^2$ is also continuous at that point. 3

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