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(April)

MATHEMATICS

(Elective/Honours)

(Geometry and Vector Calculus)

(GHS–21)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* If $ax+by$ transforms to $a'x'+b'y'$ due to the rotation of axes, then show that

$$a^2+b^2=a'^2+b'^2$$ 4

(b) Prove that the product of the perpendiculars from (p, q) to the lines represented by $ax^2+2hxy+by^2$ is

$$\frac{ap^2+2hpq+bq^2}{\sqrt{(a-b)^2+4h^2}}$$ 5

(Turn Over)

(c) Reduce the equation

$$11x^2 - 4xy + 14y^2 - 58x - 44y + 71 = 0$$

to the standard form. 6

2. (a) If two conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

intersect in four concyclic points, then show that

$$\frac{a-b}{h} = \frac{a'-b'}{h'}$$ 5

(b) Prove that the points (1, 2) and (−2, 3) are conjugate with respect to the conic

$$2x^2 + 6xy + y^2 + 4x - 2y + 8 = 0$$ 4

(c) Find the lengths of the semi-axes of the conic $7x^2 + 52xy - 32y^2 = 180$. 6

UNIT—II

3. (a) If PSP' and QSQ' be two perpendicular focal chords of the conic

$$\frac{l}{r} = 1 + e\cos\theta$$

show that

$$\frac{1}{SP \times SP'} + \frac{1}{SQ \times SQ'} = \text{constant}$$ 5

(b) Prove that the tangents at the end points of a focal chord of a parabola meet at right angles on the directrix. 5

(c) If the tangent and normal to the ellipse at a point meet the minor axis at Q and R respectively, then show that QR subtends a right angle at the focus of the ellipse. 5

4. (a) Find the asymptotes of the hyperbola

$$xy + ax + by = 0$$ 5

(b) If α and β be the eccentric angles of the extremities of a focal chord, then prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{e-1}{e+1} \text{ or } \frac{e+1}{e-1}$$ 5

(c) If the tangent $y = mx + \sqrt{a^2m^2 - b^2}$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point $(a\sec\theta, b\tan\theta)$, then prove that

$$\sin\theta = \frac{b}{am}$$ 5

UNIT—III

5. (a) Show that the two points (−2, 1, 3) and (2, 1, −1) are on the opposite sides of the plane $3x + 2y - 6z + 4 = 0$ and equidistant from it. 4

(b) Prove that the lines

$$\frac{x-2}{4}=\frac{y+1}{3}=\frac{z-3}{5}$$

and $x+2y+3z-9=0=2x-y+2z-11$ are coplanar. 5

(c) Find the equation of the sphere which passes through the points $(2, 0, 1)$ and $(0, 4, -5)$ and whose centre lies on the line $x+y+z-3=0=2x-y+2z$. 6

6. (a) Find the equation of the cone whose vertex is $(2, 2, 2)$ and the base is $z=0$, $x^2+y^2=36$. 5

(b) Prove that the equation of the right circular cylinder, whose axis is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

and radius r, is

$$(l^2+m^2+n^2)(x^2+y^2+z^2-r^2) = (lx+my+nz)^2$$ 5

(c) Find the equation of the cone, whose vertex is the origin and which passes through the curve of intersection of $x^2+y^2+z^2+2ux+d=0$, $lx+my+nz=p$ 5

UNIT—IV

7. (a) Prove that

$$|\mathbf{a}\times\mathbf{i}|^2+|\mathbf{a}\times\mathbf{j}|^2+|\mathbf{a}\times\mathbf{k}|^2=2a^2$$

where $a=|\mathbf{a}|$. 5

(b) If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non-coplanar vectors, then show that

$$[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2$$

and hence show that $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$ are non-coplanar. 5

(c) Show that the necessary and sufficient condition for the vector $\mathbf{v}(t)$ to have a constant magnitude is

$$\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}=0$$ 5

8. (a) If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors, then show that $\mathbf{r}=\mathbf{a}t^2+\mathbf{b}t+\mathbf{c}$ is the path of a particle moving with constant acceleration. 5

(b) A particle moves so that its position vector is given by $\mathbf{r}=\cos\omega t\,\mathbf{i}+\sin\omega t\,\mathbf{j}$, where ω is a constant, show that

(i) the velocity of the particle is perpendicular to $\mathbf{r}$;

(ii) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$ is a constant vector. 2+3=5

(c) If $\mathbf{u} = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$ and $\mathbf{v} = \sin t\mathbf{i} - \cos t\mathbf{j}$, then find

(i) $\frac{d}{dt}(\mathbf{u}.\mathbf{v})$;

(ii) $\frac{d}{dt}(\mathbf{u}.\mathbf{u})$. 3+2=5

UNIT—V

9. (a) If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, then prove that

$$(\text{grad}\, u).[(\text{grad}\, v) \times (\text{grad}\, w)] = 0$$ 6

(b) (i) Prove that the vector

$$\mathbf{f} = (x+3y)\mathbf{i} + (y-3z)\mathbf{j} + (x-2z)\mathbf{k}$$

is solenoidal.

(ii) Determine the constant a so that the vector

$$\mathbf{f} = (x+3y)\mathbf{i} + (y-2z)\mathbf{j} + (x+az)\mathbf{k}$$

is solenoidal. 2+2=4

(c) If $r = |\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then prove that

$$\nabla \log |\mathbf{r}| = \left(\frac{1}{r^2}\right)\mathbf{r}$$ 5

(Continued)

10. (a) Find the directional derivative of the function $f = xy + yz + zx$ in the direction of the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ at the point (3, 1, 2). 5

(b) Show that

$$\text{grad}\left(\frac{\mathbf{A}}{\mathbf{B}}\right) = \frac{\mathbf{B}(\text{grad}\mathbf{A}) - \mathbf{A}(\text{grad}\mathbf{B})}{\mathbf{B}^2}$$ 5

(c) Find the equation of the tangent plane to the surface $yz - zx + xy + 5 = 0$, at the point (1, −1, 2). 5

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